

2010 Extension 1 Solution

Question 1

$$(a) \int \frac{1}{\sqrt{4-x^2}} dx = \sin^{-1} \frac{x}{2} + C$$

$$(b) -2 \leq x \leq 2$$

$$(c) \ln(x+6) = 2 \ln x = \ln x^2$$

$$x^2 - x - 6 = 0.$$

$$(x-3)(x+2) = 0.$$

$$\therefore x = 3, -2. \text{ But } x > 0, \therefore x = 3.$$

$$(d) \frac{3}{x+2} < 4$$

$$3(x+2) < 4(x+2)^2$$

$$(x+2)(3-4(x+2)) < 0$$

$$(x+2)(-4x-5) < 0$$

$$x < -2 \text{ or } x > -\frac{5}{4}.$$

$$(e) \text{ Let } u = 1-x, du = -dx.$$

When $x = 0, u = 1$; when $x = 1, u = 0$.

$$\int_0^1 x\sqrt{1-x} dx = \int_1^0 (1-u)\sqrt{u}(-du)$$

$$= \int_0^1 (\sqrt{u} - \sqrt{u^3}) du$$

$$= \left[\frac{2\sqrt{u^3}}{3} - \frac{2\sqrt{u^5}}{5} \right]_0^1$$

$$= \frac{2}{3} - \frac{2}{5} = \frac{4}{15}.$$

$$(f) \Pr(x=2) = {}^5C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^3 = \frac{625}{3888}.$$

Question 2

$$(a) f(x) = \int \sin^2 x dx = \int \frac{1 - \cos 2x}{2} dx$$

$$= \frac{1}{2} \left(x - \frac{\sin 2x}{2} \right) + C.$$

When $x = 0, f(0) = 2, \therefore 2 = C$.

$$\therefore f(x) = \frac{x}{2} - \frac{\sin 2x}{4} + 2.$$

$$(b) (i) \frac{dM}{dt} = 35.5ke^{-kt},$$

$$\text{but } 35.5e^{-kt} = 36 - M,$$

$$\therefore \frac{dM}{dt} = k(36 - M).$$

$$(ii) t = 10, M = 20,$$

$$20 = 36 - 35.5e^{-10k}$$

$$e^{-10k} = \frac{16}{35.5} = \frac{32}{71}.$$

$$-10k = \ln \frac{32}{71}.$$

$$\therefore k = -\frac{1}{10} \ln \frac{32}{71} = 0.080.$$

(iii) As $t \rightarrow \infty, e^{-kt} \rightarrow 0, \therefore M \rightarrow 36$ tonnes.

$$(c) (i) P(3) = 0 \Rightarrow 0 = 3a + b$$

$$P(-1) = 8 \Rightarrow 8 = -a + b$$

$$\therefore a = -2, b = 6.$$

(ii) The remainder is $-2x + 6$.

$$(d) x^2 = r^2 - 36$$

$$2x \frac{dx}{dt} = 2r \frac{dr}{dt}.$$

$$x \frac{dx}{dt} = r \frac{dr}{dt}.$$

$$100x = \sqrt{x^2 + 36} \frac{dr}{dt}$$

$$\therefore \frac{dr}{dt} = \frac{100x}{\sqrt{x^2 + 36}} \text{ km/h.}$$

Question 3

$$(a) (i) \frac{5!}{2!} = 60.$$

$$(ii) 4! = 24.$$

$$(b) (i) f(x) = e^{-x^2}$$

$$f'(x) = -2xe^{-x^2}$$

$$f''(x) = -2(e^{-x^2} - 2x^2e^{-x^2}) = -2(1 - 2x^2)e^{-x^2}.$$

$$f''(x) = 0 \text{ when } 2x^2 = 1, \therefore x = \pm \frac{1}{\sqrt{2}}.$$

The points of inflexion has x -coordinates $\pm \frac{1}{\sqrt{2}}$.

(ii) $f(x)$ does not satisfy the horizontal line test.

(iii) $f: y = e^{-x^2}, x \geq 0, 0 < y \leq 1$.

$$f^{-1}: x = e^{-y^2}$$

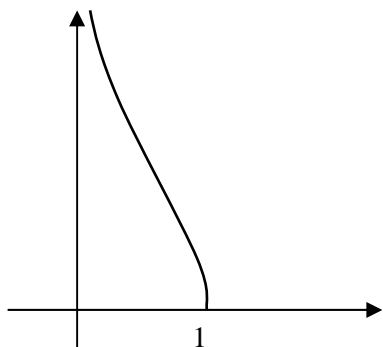
$$-y^2 = \ln x$$

$$y^2 = -\ln x = \ln \frac{1}{x}$$

$$\therefore y = +\sqrt{\ln \frac{1}{x}}, \text{ since } y \geq 0.$$

(iv) $0 < x \leq 1$.

(v)



(vi) (1) Let $g(x) = x - e^{-x^2}$,

$$\text{When } x = 0.6, g(0.6) = -0.098$$

$$\text{When } x = 0.7, g(0.7) = 0.087$$

As $g(x)$ is a continuous function, and it changes signs between $x = 0.6$ and 0.7 , it must have a zero between them.

$$(2) x_0 = \frac{0.6 + 0.7}{2} = 0.65, g(0.65) = -0.005$$

$$x_1 = \frac{0.65 + 0.7}{2} = 0.675.$$

As 0.65 and 0.675 agree to 1 decimal place, \therefore to 1 decimal place, the solution is 0.7 .

Question 4

(a) (i) $v^2 = -2(x^2 + 4x - 12) = -2(x + 6)(x - 2)$

$$v^2 = 0 \text{ when } x = -6, 2.$$

$$(ii) a = \frac{d}{dx} \left(\frac{1}{2} v^2 \right) = \frac{d}{dx} (-x^2 - 4x + 12) = -2x - 4.$$

(iii) Maximum speed occurs when $a = 0, \therefore x = -2$.

$$\therefore v^2 = 32, \therefore \text{Maximum speed} = \sqrt{32} = 4\sqrt{2}.$$

(b) (i) $2 \cos \theta + 2 \cos \left(\theta + \frac{\pi}{3} \right)$

$$= 2 \cos \theta + 2 \left(\frac{1}{2} \cos \theta - \frac{\sqrt{3}}{2} \sin \theta \right) = 3 \cos \theta - \sqrt{3} \sin \theta$$

$$= \sqrt{3} \left(\sqrt{3} \cos \theta - \sin \theta \right) = 2\sqrt{3} \left(\frac{\sqrt{3}}{2} \cos \theta - \frac{1}{2} \sin \theta \right)$$

$$= 2\sqrt{3} \left(\cos \theta \cos \frac{\pi}{6} - \sin \theta \sin \frac{\pi}{6} \right)$$

$$= 2\sqrt{3} \cos \left(\theta + \frac{\pi}{6} \right).$$

$$(ii) 2\sqrt{3} \cos \left(\theta + \frac{\pi}{6} \right) = 3$$

$$\cos \left(\theta + \frac{\pi}{6} \right) = \frac{\sqrt{3}}{2}$$

$$\theta + \frac{\pi}{6} = \pm \frac{\pi}{6} + k2\pi, k \in J.$$

$$\therefore \theta = k2\pi, -\frac{\pi}{3} + k2\pi.$$

$$\text{For } 0 < \theta < 2\pi, \theta = \frac{5\pi}{3}.$$

(c) By definition, $SP = PM = a + ap^2$.

From $y = px - ap^2$, let $x = 0, y = -ap^2, \therefore L(0, -ap^2)$

$$\therefore SL = a - (-ap^2) = a + ap^2.$$

$$M(2ap, -a), \therefore LM = \sqrt{(2ap - 0)^2 + (-a + ap^2)^2}$$

$$= \sqrt{4a^2 p^2 + a^2 - 2a^2 p^2 + a^2 p^4}$$

$$= \sqrt{a^2 + 2a^2 p^2 + a^2 p^4} = \sqrt{(a + ap^2)^2} = a + ap^2.$$

$\therefore SLMP$ is a rhombus (4 sides equal)

Question 5

(a) (i) $\tan 20^\circ = \frac{PL}{AP} = \frac{1}{AP}, \therefore AP = \frac{1}{\tan 20^\circ}$.

$$\tan 3^\circ = \frac{PT}{AP} = PT \tan 20^\circ. \quad (1)$$

$$\tan 30^\circ = \frac{PT}{BP}, \therefore \frac{1}{\sqrt{3}} = \frac{PT}{BP}. \quad (2)$$

$$\frac{(1)}{(2)} \Rightarrow \sqrt{3} \tan 3^\circ = BP \tan 20^\circ.$$

$$\therefore BP = \frac{\sqrt{3} \tan 3^\circ}{\tan 20^\circ}.$$

$$(ii) AB = AP - BP = \frac{1 - \sqrt{3} \tan 3^\circ}{\tan 20^\circ}.$$

$$(b) (i) f'(x) = \frac{1}{1+x^2} + \frac{1}{1+\frac{1}{x^2}} \times \frac{-1}{x^2}$$

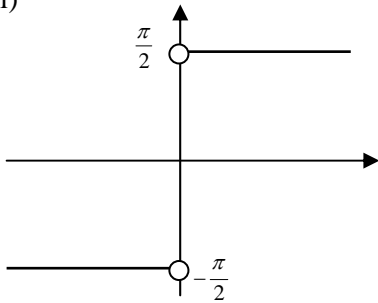
$$= \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0.$$

$\therefore f(x)$ is constant.

$$\text{Substitute } x=1, f(1) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

$$\therefore \text{For } x > 0, f(x) = \frac{\pi}{2}.$$

(ii)



(c) (i) Exterior angle in a Δ is the sum of the two opposite interior angles.

(ii) The angle between a tangent and a chord is equal to any angle in the alternate segment.

(iii) $\angle ABD = \angle TAC$ (same theorem in (ii)),

$\therefore \angle XDB = \angle CAD$ (from (i) and (ii)),

But $\angle XAD = \angle XDB$ (same theorem in (ii)),

$\therefore \angle XAD = \angle CAD$.

$\therefore AD$ bisects $\angle BAC$.

Question 6

$$(a) (i) \cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$= \cos A \cos B \left(1 + \frac{\sin A \sin B}{\cos A \cos B} \right)$$

$$= \cos A \cos B (1 + \tan A \tan B)$$

(ii) If $\tan A \tan B = -1$, $\cos(A-B) = 0$

$$\therefore A-B = \pm \frac{\pi}{2} + k2\pi.$$

$$\text{But } 0 < B < \frac{\pi}{2} \text{ and } B < A < \pi, \therefore A-B = \frac{\pi}{2}.$$

(b) (i) Substitute $x=d, y=h$,

$$d = vt \cos \theta, \therefore t = \frac{d}{v \cos \theta}.$$

$$h = vt \sin \theta - 5t^2 = v \frac{d}{v \cos \theta} \sin \theta - 5 \frac{d^2}{v^2 \cos^2 \theta}$$

$$= d \tan \theta - \frac{5d^2}{v^2 \cos^2 \theta}.$$

$$\therefore \frac{5d^2}{v^2 \cos^2 \theta} = d \tan \theta - h,$$

$$v^2 \cos^2 \theta = \frac{5d^2}{d \tan \theta - h}.$$

But $h = d \tan \alpha$,

$$\therefore v^2 \cos^2 \theta = \frac{5d^2}{d \tan \theta - d \tan \alpha} = \frac{5d}{\tan \theta - \tan \alpha}$$

$$v^2 = \frac{5d}{\cos^2 \theta \tan \theta - \cos^2 \theta \tan \alpha}$$

$$= \frac{5d}{\cos \theta \sin \theta - \cos^2 \theta \tan \alpha}.$$

(ii) (1) When $\theta \rightarrow \alpha, v^2 \rightarrow \frac{5d}{\cos \alpha \sin \alpha - \cos \alpha \sin \alpha} \rightarrow \frac{5d}{0} \rightarrow \infty, \therefore v \rightarrow \infty$.

(2) When $\theta \rightarrow \frac{\pi}{2}, \cos \frac{\pi}{2} \rightarrow 0, v^2 \rightarrow \frac{5d}{0} \rightarrow \infty, \therefore v \rightarrow \infty$.

(iii) $F'(\theta) = -\sin^2 \theta + \cos^2 \theta + 2 \cos \theta \sin \theta \tan \alpha$

$$= \cos 2\theta + \sin 2\theta \tan \alpha$$

$$= 0 \text{ when } \cos 2\theta = -\sin 2\theta \tan \alpha$$

$$\therefore \tan 2\theta \tan \alpha = -1.$$

(iv) From (a) (ii), $2\theta - \alpha = \frac{\pi}{2}, \therefore \theta = \frac{\alpha}{2} + \frac{\pi}{4}$.

(v) $v^2 = \frac{5d}{F(\theta)}, \therefore \frac{d}{d\theta}(v^2) = \frac{-5d \times F'(\theta)}{F^2(\theta)}$.

$$\frac{d}{d\theta}(v^2) = 0 \text{ when } F'(\theta) = 0, \text{ i.e. } \theta = \frac{\alpha}{2} + \frac{\pi}{4}.$$

As v^2 is a continuous function for $\alpha < \theta < \frac{\pi}{2}$, and v^2

$\rightarrow \infty$ for $\theta \rightarrow \alpha$ and $\theta \rightarrow \frac{\pi}{2}$, v^2 has a minimum value.

Further, as $\theta = \frac{\alpha}{2} + \frac{\pi}{4}$ is the unique value for which

$$F'(\theta) = 0, \alpha < \theta < \frac{\pi}{2}, v^2 \text{ is minimum when } \theta = \frac{\alpha}{2} + \frac{\pi}{4}$$

Question 7

(a) Let $n = 1, 47 + 53 \times 147^0 = 47 + 53 = 100$, which is divisible by 100.

Assume $47^n + 53 \times 147^{n-1}$ is divisible by 100,

let $47^n + 53 \times 147^{n-1} = 100M$, where M is an integer.

$$\therefore 47^n = 100M - 53 \times 147^{n-1}.$$

$$\begin{aligned} 47^{n+1} + 53 \times 147^n &= 47(100M - 53 \times 147^{n-1}) + 53 \times 147^n \\ &= 4700M + 53 \times 147^{n-1}(-47 + 147) \\ &= 4700M + 5300 \times 147^{n-1} \\ &= 100(47M + 53 \times 147^{n-1}). \end{aligned}$$

$\therefore 47^{n+1} + 53 \times 147^n$ is divisible by 100.

By the Principle of Induction, $47^n + 53 \times 147^{n-1}$ is divisible by 100 for all $n \geq 1$.

(b) (i) Put $x = 1$, $2^n = \sum_{k=0}^n \binom{n}{k}$.

(ii) Put $n = 100$, $2^{100} = \sum_{k=0}^{100} \binom{100}{k}$.

(iii) Differentiating both sides of

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n \text{ gives}$$

$$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + \dots + n\binom{n}{n}x^{n-1}.$$

Put $x = 1$, $n2^{n-1} = k \sum_{k=1}^n \binom{n}{k}$.

(c) (i) Since there are two colours, selecting 1 ball has 2 ways (either R or B), selecting 2 balls has 3 ways (2R, 1R1B, 2B), selecting 3 balls has 4 ways (3R, 1R2B, 2R1B, 3B), and so on.

\therefore The number of ways of selecting r balls is the

number of terms of $\binom{r}{0}, \binom{r}{1}, \binom{r}{2}, \dots, \binom{r}{r}$, i.e. $r+1$

ways.

(ii) Selecting $n-r$ balls from n different balls has

$\binom{n}{n-r}$ ways. But $\binom{n}{n-r} = \binom{n}{r}$, due to symmetry.

(iii) In part (c) (ii), the number of ways of

selecting $(n-r)$ balls from n different balls is $\binom{n}{r}$,

and in part (i), the number of ways of selecting r balls from n identical red balls and n identical blue balls is $r+1$.

\therefore The total number of ways of selecting $n, n-1, \dots, 0$

white balls is $\binom{n}{0} + 2\binom{n}{1} + 3\binom{n}{2} + \dots + (n+1)\binom{n}{n}$,

given that once $n-r$ white balls are chosen, there is $r+1$ possible number of ways of choosing r balls from the identical red or blue balls.

As $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n$, from part (b) (i),

$\binom{n}{1} + 2\binom{n}{2} + \dots + n\binom{n}{n} = n2^{n-1}$, from part (b) (iii),

\therefore the total $= 2^{n-1} + n2^{n-1} = (n+2)2^{n-1}$.

Note: The question was worded poorly. What they meant was n identical red balls, n identical blue balls and n labelled white balls.