Multiple Choice

1) (D) $(2\times1+1)+(2\times2+1)+...+(2\times20+1)$

2) (B) $P(2)=2\times8-10\times4+6\times2+2=-10$

3) (A) $\tan x=\tan(2x-x)$

4) (C) $360°-(180°+40°)=140°, \therefore$ the reflex angle $=220°$

5) (A) $=\int\frac{1-\cos4x}{2}dx$

6) (D) $(2\sin x-1)(\sin x-3)=0, \sin x=\frac{1}{2},$

$\therefore x=n\pi+(-1)^n\frac{\pi}{6}$

7) (C) $\dot{x}=20\cos4t-48\sin4t.$

Amplitude $=\sqrt{20^2+48^2}=52$

8) (B), At least $1={}^{18}C_{11}-{}^{15}C_{11}=30459$

9) (A) $f(1)<1, f'(x)>1$ and $f''(1)<0$ (concave down)

10) (A) $p'(x)=3ax^2+2bx+c, p''(x)=6ax+2b.$

POI occurs at $x=-\frac{b}{3a}<0$, since $a>0, b>0$

Question 11

(a) $f: y=x^3-2$

$f^{-1}: x=y^3-2, \therefore y=\sqrt[3]{x+2}$

(b) $u=x-4, du=dx$

$$\int(u+4)\sqrt{u}du=\int\left(u^{\frac{3}{2}}+4u^{\frac{1}{2}}\right)du=\frac{2u^{\frac{5}{2}}}{5}+\frac{8u^{\frac{3}{2}}}{3}+C$$

$$=\frac{2\sqrt{(x-4)^5}}{5}+\frac{8\sqrt{(x-4)^3}}{3}+C$$

(c) $\frac{6}{1+4x^2}$

(d) $\lim_{x\to0}\frac{2\sin x\cos x}{3x}=\lim_{x\to0}\frac{\sin2x}{3x}=\frac{2}{3}\lim_{x\to0}\frac{\sin2x}{2x}=\frac{2}{3}$

(e) $\frac{3}{2x+5}>x$

$3(2x+5)>x(2x+5)^2, x\neq-\frac{5}{2}$

$(2x+5)(2x^2+5x-3)<0$

$(2x+5)(2x-1)(x+3)<0$

$\therefore x<-3, -\frac{5}{2}<x<\frac{1}{2}$

(f) (i) $P(\text{exactly once in 3 throws})={}^3C_1\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)^2=\frac{36}{125}$

(ii) $P(\text{at least 2 in 6 throws})=1-P(x=0\text{ or }1)$

$=1-\left(\frac{2}{5}\right)^6-{}^6C_1\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)^5=\frac{2997}{3125}$

Question 12

(a)(i) By similar triangles, $\frac{r}{5}=\frac{h}{20}, \therefore r=\frac{h}{4}$

(ii) $v=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi h\frac{h^2}{16}=\frac{\pi}{48}h^3$

$\frac{dv}{dh}=\frac{\pi}{48}\times 3h^2=\frac{\pi}{16}h^2$

(iii) Given $\frac{dA}{dt}=-0.04\text{ cm}^2\text{s}^{-1}, A=\pi r^2=\frac{\pi}{16}h^2$

$\therefore \frac{dA}{dh}=\frac{\pi h}{8}$

$\frac{dh}{dt}=\frac{dh}{dA}\frac{dA}{dt}=\frac{8}{\pi h}\times -0.04=\frac{-0.32}{\pi h}$

(iv) $\frac{dv}{dt}=\frac{dv}{dh}\frac{dh}{dt}=\frac{\pi}{16}h^2\times\frac{-0.32}{\pi h}=-0.02h$

When $h=10, \frac{dv}{dt}=-0.2\text{ cm}^3\text{s}^{-1}$

(b) (i) $\frac{dx}{dt}=ky=k(500-x)$

When $x=0, \frac{dx}{dt}=2=500k, \therefore k=\frac{2}{500}=0.004$

$\therefore \frac{dx}{dt}=0.004(500-x)$

(ii) $x=500-Ae^{-0.004t}$, i.e. $Ae^{-0.004t}=500-x$

$\frac{dx}{dt}=0.004Ae^{-0.004t}=0.004(500-x)$

$\therefore x=500-Ae^{-0.004t}$ satisfies the equation.

When $t=0, x=0, \therefore A=500.$

(c) (i) For $y=\tan x, y'=\sec^2 x$. At $x=\alpha, m_1=\sec^2\alpha$

For $y=\cos x, y'=-\sin x$. At $x=\alpha, m_2=-\sin\alpha$

When they meet at $x=\alpha$, $\tan\alpha=\cos\alpha$,

$\frac{\sin\alpha}{\cos\alpha}=\cos\alpha$

$\cos^2\alpha=\sin\alpha$.

$m_1 m_2=-\sec^2\alpha\sin\alpha=-\frac{\sin\alpha}{\cos^2\alpha}=-1, \therefore$ the tangents are perpendicular.

(ii) $f(x)=\tan x-\cos x$,

$f'(x)=\sec^2 x+\sin x$

$x_2=1-\frac{f(1)}{f'(1)}=0.76$

Question 13

(a) (i) High tide $=9$, low tide $=1, \therefore$ amplitude $=4$.

Period $=\frac{25}{2}$ hours, $\therefore w=\frac{2\pi}{\text{Period}}=\frac{4\pi}{25}$.

By choosing the origin when the tide is 5 m, and when $t=0$, it's the high tide, $x=5+4\cos\left(\frac{4\pi}{25}t\right)$.

(ii) $\frac{dx}{dt}=-\frac{16\pi}{25}\sin\left(\frac{4\pi}{25}t\right)$

$\frac{dx}{dt}$ is maximum and positive when $\sin\left(\frac{4\pi}{25}t\right)=-1$.

$\frac{4\pi}{25}t=\frac{3\pi}{2}$

$t=\frac{75}{8}$, taking the earliest time

$\therefore$ It's $\frac{75}{8}$ hours after 2 am, i.e. 11:22:30 am.

(b) (i) $\dot{y}=u\sin\theta-10t$

$\dot{y}=0$ when $t=\frac{u\sin\theta}{10}$

$\therefore y=\frac{u^2\sin^2\theta}{10}-5\frac{u^2\sin^2\theta}{100}=\frac{u^2\sin^2\theta}{20}$.

$\therefore$ The greatest height is $\frac{u^2\sin^2\theta}{20}$.

(ii) $u=30, \theta=30°, \therefore$ The greatest height above the projection is $\frac{30^2\sin^2 30°}{20}=\frac{45}{4}$.

$\therefore$ The height above the ground is $\frac{45}{4}+20=\frac{125}{4}$ m

(iii) For the ball to rebound, it follows the reversed path. Taking $y=-5t^2=\frac{-125}{4}$, since $\dot{y}=0$

$t^2=\frac{125}{20}=\frac{25}{4}, \therefore t=\frac{5}{2}$ s.

(iv) $x=ut=10\times\frac{5}{2}=25$ m.

(c) (i) $\angle ACB=\angle ADB=90°$ (semi-circle angles)

$\therefore CMDE$ is a cyclic quad because opposite angles are supplementary.

(ii) In ΔMDE and ΔAEF,

$\angle MED=\angle AEF$ (vertically opposite angles)

$\angle DME=\angle EAF$ (both $=90°-\angle DBA$, angle sum in a Δ)

$\therefore \Delta MDE \,|||\, \Delta AEF$ (AA)

$\therefore MF\perp AB$ (corresponding angles in similar Δs)

Question 14

(a) (i) RHS $=4n^3+14n^2+9n+4n^2+14n+9$

$=4n^3+18n^2+23n+9$

(ii) Let $n=1$, LHS $=1\times3=3$, RHS $=\frac{1}{3}\times1\times9=3$

The statement is true for $n=1$

Assume $1\times3+3\times5+...+(2n-1)(2n+1)=$

$\frac{1}{3}n\left(4n^2+6n-1\right)$, for some value of n.

Required to prove

$1\times3+3\times5+...+(2n-1)(2n+1)+(2n+1)(2n+3)$

$=\frac{1}{3}(n+1)\left(4(n+1)^2+6(n+1)-1\right)$

RHS $=\frac{1}{3}(n+1)\left(4n^2+8n+4+6n+6-1\right)$

$=\frac{1}{3}(n+1)\left(4n^2+14n+9\right)$

$=\frac{1}{3}\left(4n^3+18n^2+23n+9\right)$.

LHS $=\frac{1}{3}n\left(4n^2+6n-1\right)+(2n+1)(2n+3)$

$=\frac{1}{3}\left(4n^3+6n^2-n\right)+\left(4n^2+8n+3\right)$

$=\frac{1}{3}\left(4n^3+6n^2-n+12n^2+24n+9\right)$

$=\frac{1}{3}\left(4n^3+18n^2+23n+9\right)=$ RHS.

By the principle of Induction, it's true for all $n\geq1$.

(b) (i) $(1+x)^n=\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+...+\binom{n}{n}x^n$ (1)

Let $x=1, 2^n=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+...+\binom{n}{n}$.

(ii) Differentiate both sides of (1) wrt x,

$n(1+x)^{n-1}=\binom{n}{1}+2\binom{n}{2}x+3\binom{n}{3}x^2+...+n\binom{n}{n}x^{n-1}$

Let $x=1, n2^{n-1}=\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+...+n\binom{n}{n}$.

(iii) Multiplying the result of (i) by n gives

$$n2^n = n\left(\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n}\right)=\sum_{r=0}^{n}\binom{n}{r}n$$

$$\therefore n2^n - n = \sum_{r=1}^{n}\binom{n}{r}n \qquad (2)$$

Multiplying the result of (ii) by 2 gives

$$2n2^{n-1} = 2\left(\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+\ldots+n\binom{n}{n}\right)$$

$$=\sum_{r=1}^{n}\binom{n}{r}2r \qquad (3)$$

$(3)-(2)$ gives

$$\sum_{r=1}^{n}\binom{n}{r}(2r-n) = 2n2^{n-1}-\left(n2^n - n\right)$$

$$= n2^n - \left(n2^n - n\right) = n$$

(c) (i) The equation of the tangent is $y = tx - at^2$

When $y=-a, tx = at^2 - a, \therefore x = at - \frac{a}{t}$.

$$\therefore D\left(at-\frac{a}{t}, -a\right)$$

(ii) The equation of the normal

$$y - at^2 = -\frac{1}{t}(x-2at)$$

Sub $x = at - \frac{a}{t}$ gives $y - at^2 = \frac{1}{t}\left(at+\frac{a}{t}\right)$

$$y = a + \frac{a}{t^2} + at^2.$$

$$R\begin{cases} x = a\left(t-\frac{1}{t}\right) \\ y - a = a\left(t^2+\frac{1}{t^2}\right)\end{cases}$$

Use $\left(t-\frac{1}{t}\right)^2 = t^2 + \frac{1}{t^2} - 2$ gives

$$\frac{x^2}{a^2} = \frac{y-a}{a} - 2$$

$$\frac{x^2}{a^2} = \frac{y}{a} - 3$$

$\therefore$ The locus is the parabola $x^2 = ay - 3a^2$

(iii) The focal length is $\frac{a}{4}$

(iv) For P_2, $y = \frac{x^2+3a^2}{a}$

$$\therefore y' = \frac{2x}{a}$$

$\therefore$ The gradient of the normal to P_2 at R where

$x = a\left(t-\frac{1}{t}\right)$ is $m = -\frac{a}{2x} = -\frac{1}{2\left(t-\frac{1}{t}\right)} = -\frac{t}{2(t^2-1)}$

The 2 normals are identical when $-\frac{t}{2(t^2-1)} = -\frac{1}{t}$

$$t^2 = 2(t^2-1)$$

$$t^2 = 2$$

$$t = \pm\sqrt{2}$$